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On the Petrov classification of gravitational fields

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Abstract. An intuitive directional symmetry for a null direction with respect to a given local Lorentz observer is suggested. It is shown that the existence of a null direction with such a symmetry with respect to the given Lorentz observer implies that the null direction is a Debever–Penrose direction. A weak converse is also given. In the vacuum case it is shown that the existence of such a null direction is equivalent to the space–time being algebraically special in the Petrov classification and having that null direction as a repeated principal null direction. The results are generalized to non-vacuum space–times. Finally it is suggested how a vacuum space–time might be classified according to the type of symmetries it possesses.

1. Introduction: the Petrov classification

A convenient means of classifying gravitational fields in general relativity theory is provided by the canonical forms of Petrov. This procedure is a systematic study of the various algebraic forms (types) which may be assumed by the Weyl tensor, C_{abcd} of the space–time manifold‡. On denoting the Riemann tensor, Ricci tensor and Ricci scalar by R_{abcd}, R_{ab} and R respectively, one has $R_{ab} = R_{cabd}g^{cd}$, $R = R_{ab}g^{ab}$ and

$$C_{abcd} = R_{abcd}+R_{c[a}g_{b]d}+R_{d[b}g_{a]c}+\tfrac{1}{3}Rg_{c[b}g_{a]d} \tag{1.1}$$

where g_{ab} is the space–time metric tensor. *In vacuo*, one has the equivalent conditions $R_{ab} = 0 \Leftrightarrow C_{abcd} = R_{abcd}$.

Of the many different approaches to the Petrov classification of the Weyl tensor (Riemann tensor *in vacuo*) the elegant Bel criteria (Bel 1962) will be most useful here. One of Bel's results is that a Weyl tensor is algebraically special in the Petrov classification if and only if there exists a (necessarily null) real principal direction $l^a \neq 0$ such that $l^a l^c \overset{+}{C}_{abc[d}l_{e]} = 0 \; (C_{abcd} \neq 0)$ where

$$\overset{+}{C}_{abcd} = C_{abcd}+\mathrm{i}\overset{*}{C}_{abcd}$$

$$\overset{*}{C}_{abcd} = \tfrac{1}{2}\sqrt{-g}\,\eta_{abmn}C^{mn}{}_{cd} = \tfrac{1}{2}\sqrt{-g}\,\eta_{mncd}C_{ab}{}^{mn}.$$

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‡ Small Latin indices take the values 0, 1, 2, 3. Round and square brackets denote the usual symmetrization and antisymmetrization respectively. The metric signature will be taken as +2, the symbol g denotes $\det(g_{ab})$ and η_{abcd} will represent the Levi-Civita alternating symbol. A comma will denote a partial derivative and a semicolon a covariant derivative. An asterisk will denote the usual duality operator.

Algebraically special space–times can also be characterized by the single real condition $l^a l^c C_{abc[d}l_{e]} = 0$ if l^a is given to be null, or by the Bel equations

$$l^a l^c C_{abcd} = \alpha l_b l_d \qquad l^a l^c \overset{*}{C}_{abcd} = \beta l_b l_d. \tag{1.2}$$

In Petrov type D space–times there are two choices of the principal null direction l^a. For other algebraically special space–times the principal direction l^a is uniquely determined to within a sign. In Petrov type III or N space–times $\alpha = \beta = 0$. Finally, a real direction l^a is said to be a Debever–Penrose direction if $l^b l^c l_{[m}\overset{+}{C}_{a]bc[d}l_{n]} = 0$. Such a direction is necessarily null. This condition can be replaced by the single real condition $l^b l^c l_{[m}C_{a]bc[d}l_{n]} = 0$ if l^a is given to be null†.

The object of the present paper is to give a simple interpretation of algebraically special space–times, especially those space–times which represent vacuum gravitational fields.

One final definition is required. The symbol $K(n, n')$ will denote the riemannian curvature of the elementary two-space spanned by the vectors n^a and n'^a. Thus

$$K(n, n') = \frac{-R_{abcd}n^a n^c n'^b n'^d}{2g_{a[c}g_{d]b}n^a n^c n'^b n'^d} \qquad (g_{a[c}g_{d]b}n^a n^c n'^b n'^d \neq 0). \tag{1.3}$$

This definition, apart from an unimportant sign, agrees with that given by Eisenhart (1966). For the remainder of the paper only the case $C_{abcd} \neq 0$ is considered.

2. Riemann symmetries

Suppose l^a is a real null vector field defined in a region of a space–time manifold. Let P be a point of this region at which a local Lorentz observer O is introduced such that with respect to O at P, $l^a = \delta^a_1 + \delta^a_0$. Consider the 'cone' of vectors $\{r^a, A\}$ for O at P for a fixed non-zero parameter A and variable parameters μ and λ given by

$$\begin{aligned} &\{r^a, A\} = \{r^a : r^a = Ax^a + \mu y^a + \lambda z^a\} \\ &x^a = \delta^a_1, \qquad y^a = \delta^a_2, \qquad z^a = \delta^a_3, \qquad \mu^2 + \lambda^2 = 1 \\ &l^a y_a = l^a z_a = y^a z_a = x^a z_a = x^a y_a = l^a l_a = 0. \end{aligned} \tag{2.1}$$

We shall call l^a Riemann symmetric for O at P if $K(l, r)$ is independent of μ and λ at P and completely Riemann symmetric for O at P if $K(l, r)$ is independent of μ, λ and A at P. Thus l^a is Riemann symmetric for O at P if and only if

$$K(l, r) = A^{-2}R_{abcd}l^a l^c r^b r^d, \qquad \text{is independent of } \mu \text{ and } \lambda \text{ at P.} \tag{2.2}$$

By defining $F_{bd} = R_{abcd}l^a l^c = F_{db}$, the symmetries of the Riemann tensor yield $F_{ab}l^b = 0$. Then by using r^a given in (2.1) and the condition (2.2) (remembering the restriction $\mu^2 + \lambda^2 = 1$) one easily finds in the frame O that $F_{12} = F_{13} = F_{23} = F_{20} = F_{30} = 0$, $F_{22} = F_{33}$, $F_{11} = -F_{01}$ and $F_{10} = -F_{00}$. Hence one may represent F_{ab} in terms of two invariants F and G in the canonical form

$$F_{bd} = R_{abcd}l^a l^c = Gl_b l_d + F(y_b y_d + z_b z_d). \tag{2.3}$$

† As well as Bel's paper, a thorough discussion of the Petrov forms can be found in Sachs' paper (Sachs 1961). One recalls that the Petrov types N, III, II and D correspond to the types 3b, 3a, 2b and 2a respectively in Bel's notation.

Equation (2.3) is the necessary and sufficient condition that l^a be Riemann symmetric for O at P. In fact it is clearly the necessary and sufficient condition that l^a be Riemann symmetric for O′ at P where O′ is any member of a two-parameter set of Lorentz observers obtained from O by a combination of a Lorentz boost in the x^a direction and a spacelike rotation in the yz plane. By using the completeness relation $g_{ab} = 2l_{(a}m_{b)}+y_a y_b+z_a z_b$, where m^a is a null vector uniquely determined by the conditions $m^a y^a = m^a z_a = m^a m_a = 0$, $l^a m_a = 1$, together with the equations (1.1) and (2.3) one easily finds the condition $l^b l^c l_{[m}C_{a]bc[d}l_{n]} = 0$. Thus l^a is a Debever–Penrose vector. A weak converse can also be proved here. If l^a is a Debever–Penrose vector at P, then it easily follows that there exists a vector q^a at P with $l^a q_a = 0$ and $l^a l^c C_{abcd} = 2l_{(b}q_{d)}$. Next, suppose that $R_{ab}l^a l^b \neq 0$ at P. Then the vectors l^a and $p^a = q^a+\frac{1}{2}R^a_b\, l^b$ determine a null direction l'^a distinct from l^a and unique up to sign lying in the elementary two-space spanned by l^a and p^a. It is then easy to show that the null direction l^a is Riemann symmetric at P for any member of the two-parameter set of Lorentz observers whose spacelike elementary two-spaces spanned by y^a and z^a in the sense of (2.1) are orthogonal to both l^a and l'^a. The proof consists of constructing $R_{abcd}l^a l^c$ from (1.1) using the above Debever–Penrose condition on C_{abcd} and then comparing with (2.3). Collecting these results together one has the following theorem.

Theorem 1.

If a null direction is Riemann symmetric for some observer at a point P, then it is a Debever–Penrose direction at P. Conversely if l^a is a Debever–Penrose direction at P and $R_{ab}l^a l^b \neq 0$ at P, then the Weyl tensor and Ricci tensor determine a two-parameter set of Lorentz observers for whom l^a is Riemann symmetric at P.

If $R_{ab}l^a l^b = 0$ at P it will become apparent (see the remarks following (2.4)) that either l^a is completely Riemann symmetric at P or it is not Riemann symmetric for any O at P.

Next, from (2.3) it follows that

$$K(l,r) = A^{-2}R_{abcd}l^a l^c r^b r^d = G+A^{-2}F \tag{2.4}$$

whilst a simple contraction of (2.3) yields $R_{ab}l^a l^b = -2F$. Thus if l^a is Riemann symmetric for O at P, it is completely Riemann symmetric for O at P if and only if $F = -\frac{1}{2}R_{ab}l^a l^b = 0$.

So far, only the Riemann symmetry properties of l^a with respect to a two-parameter set of Lorentz observers have been considered. All these Lorentz observers have the same elementary spacelike two-surface spanned by their respective y and z axes. In order to extend this concept, the direction l^a shall be called Riemann symmetric at P if and only if $K(l,r)$ is independent of μ and λ for all Lorentz observers at P for which $l^a \propto \delta^a_1+\delta^a_0$ and completely Riemann symmetric at P if and only if $K(l,r)$ is independent of μ, λ and A for all such Lorentz observers at P. Then l^a is Riemann symmetric at P if (2.2) holds for each member of a four-parameter collection of Lorentz observers connected by the well known 'null rotation' subgroup of the homogeneous Lorentz group. Two of these parameters are of course the two mentioned above. The other two-parameter set of observers have different spacelike elementary two-surfaces spanned by y and z. Mathematically this extra condition means that (2.2) must remain 'invariant' under the two-parameter null rotations given by†

$$y^a \to y'^a = y^a+al^a \qquad z^a \to z'^a = z^a+bl^a. \tag{2.5}$$

† A discussion of null rotations can be found in the paper by Sachs (1961).

From (2.3) this clearly implies that $F = -\frac{1}{2}R_{ab}l^a l^b = 0$. It then follows that if l^a is Riemann symmetric at P then it is completely Riemann symmetric at P. On collecting these results together one has the following theorem.

Theorem 2.

A null direction l^a is Riemann symmetric for O at P if and only if (2.3) holds at P. This null direction is completely Riemann symmetric for O at P if and only if (2.3) holds with $F = 0$ at P, which in turn holds if and only if l^a is Riemann symmetric, equivalently completely Riemann symmetric, at P.

Finally, one can easily show from (1.1) that any two of the following conditions for a null direction l^a imply the third:

$$l^a l^c C_{abc[d}l_{e]} = 0, \qquad l^a l^c R_{abc[d}l_{e]} = 0, \qquad R_{ab}l^b = \sigma l_a \tag{2.6}$$

where σ is an invariant. Then since for a Riemann symmetric null direction l^a one has from (2.3) with $F = 0$

$$R_{abcd}l^a l^c = G l_b l_d \qquad (\Rightarrow l^a l^c R_{abc[d}l_{e]} = 0) \tag{2.7}$$

one easily obtains the following theorem.

Theorem 3.

If a null direction l^a is Riemann symmetric at P, then the Weyl tensor is algebraically special in the Petrov classification at P with repeated principal null direction l^a if and only if l^a is a Ricci eigenvector at P.

For a vacuum field this latter condition is trivially satisfied whilst for a null Maxwell–Einstein field with Ricci tensor proportional to $l_a l_b$, the null direction l^a is necessarily Riemann symmetric since a null Maxwell field either has $C_{abcd} = 0$ or is algebraically special with repeated principal null direction l^a (Goldberg and Sachs 1962).

3. Vacuum gravitational fields

For vacuum gravitational fields one can utilize the notion of Riemann symmetries to obtain the following restatement of Petrov's classification. If it is agreed to call a congruence of null curves Riemann symmetric in a four-dimensional region R if the tangent vector at a point P of the congruence is Riemann symmetric for all points P of R, then one has the following results. A non-flat four-dimensional region R of a vacuum space–time is algebraically special in the Petrov classification if and only if it admits a Riemann symmetric null congruence l^a. This direction is a repeated principal null direction. The region R is Petrov type N or III if and only if $K(l, r) = 0$ throughout R, whence the congruence l^a is necessarily unique. Further, if l^a is unique and $K(l, r) \neq 0$ then R is of Petrov type II and, conversely, if R is of Petrov type II then a single null Riemann symmetric congruence is admitted with $K(l, r) \neq 0$ in general at all points of R except perhaps on certain subspaces of R. Finally if R admits two distinct Riemann symmetric null congruences, l^a and m^a then it is Petrov type D and conversely. Necessarily $K(l, r) = K(m, r')$† and their common value is non-zero in general except perhaps on certain subspaces of R.

† Here, r'^a is given with respect to m^a in a similar way to which r^a was given with respect to l^a.

The proof of these statements is mostly contained in § 2. It is only required to show that for type II and D vacuum space–times, $\alpha = 0 \Rightarrow \beta = 0$ in the notation of (1.2). To show this one firstly substitutes the canonical form for a type II vacuum Riemann tensor (Bel 1962, Debever 1959) into the vacuum Bianchi identity $(R_{abcd} + \mathrm{i}\overset{*}{R}_{abcd})^{;d} = 0$ and contracts with $l^a l^c$. Then by using (1.2) together with some calculation one finds

$$\alpha_{,a}l^a + 3\alpha\theta - \tfrac{3}{2}\beta\omega = \beta_{,a}l^a + 3\beta\theta + \tfrac{3}{2}\alpha\omega = 0 \tag{3.1}$$

where θ and ω represent the expansion and twist of l^a respectively†. By a similar argument one can obtain expressions like (3.1) for both repeated principal null directions of a vacuum type D field. Suppose now that one has a type II or D field with $\alpha = 0$, $\beta \neq 0$. Then by (3.1) the principal null direction(s) is(are) twist-free. Also, by the Goldberg–Sachs theorem (Goldberg and Sachs 1962) the principal null direction(s) must also be shear-free since R_{abcd} is algebraically special. It then follows that the metric in R can be reduced to one of the Robinson–Trautman metrics (Robinson and Trautman 1962) or one of the Kundt metrics (Kundt 1961). However, in the former case, $\beta = 0$ always. In the latter case the line element can be written in the form

$$\mathrm{d}s^2 = P^2(\mathrm{d}x^2 + \mathrm{d}y^2) + 2m_a\,\mathrm{d}x^a\,\mathrm{d}u \tag{3.2}$$

where $m_3 = 1$, $x^4 \equiv u =$ constant, represents hypersurfaces orthogonal to $l_a = u_{,a}$, $x^3 = v$ is an affine parameter along the paths of l^a and $P_{,3} = 0 < P$. The condition $\alpha = 0$ when applied to (3.2) yields

$$\left(\frac{\partial^2}{\partial x^{1^2}} + \frac{\partial^2}{\partial x^{2^2}}\right)(\ln P) = 0.$$

This condition however implies the existence of a coordinate transformation reducing the metric to the form (3.2) with $P = 1$ (Kundt 1961) which, Kundt then demonstrated, implied that the space–time was type III or N. This completes the proof.

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† For a discussion of the optical scalars expansion, twist, and shear see Sachs (1961) and Pirani (1964).